Theorem (Brjuno, Yoccoz) $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. $(\mathcal{L} = \mathcal{B})$.

Then $\forall f: (\mathbb{C},0) \circlearrowleft$, $f'(0) = e^{2\pi i \alpha}$ is (analytically) linearizable if and only if α satisfies the Brjuno condition.

($\Leftarrow$ Brjuno, by studying the growth of $|\psi_n|$, coefficients of ψ so that

$f \circ \psi(z) = \psi(\lambda z)$. $\lambda = e^{2\pi i \alpha}$

$\Rightarrow$ Yoccoz, by showing that if the Brjuno condition, then $\lambda z + z^2$ is not lin.)

We will not prove this result, but we see how Diophantine conditions play their role showing that $\mathcal{D}(2) \subseteq \mathcal{L}$.

Theorem (Herman): $\mathcal{D}(2) \subseteq \mathcal{L}$: let $\alpha \in \mathcal{D}(2)$; and $\lambda = e^{2\pi i \alpha}$. Then any map $f(z) = \lambda z (1 + o(1))$ is linearizable, $\exists! \, \Psi(z) = z(1+o(1))$ so that $f \circ \Psi(z) = \Psi(\lambda z)$ $(*)$

Rem: this is the inverse of the conjugacy considered until now; for some reasons, it works better this way. Also for the proof of Brjuno theorem.

Proof: by differentiating and taking the logarithm of $(*)$, we get:

$$\log f'(\Psi(z)) + \log \Psi'(z) = \log \lambda + \log \Psi'(\lambda z), \text{ or equivalently,}$$

$$\log \Psi'(\lambda z) - \log \Psi'(z) = \log \frac{f'(\Psi(z))}{\lambda}.$$

The idea is to decompose the problem considering the three transformations:

$$F_1(\varphi_0)(z) := \log \frac{f'(\varphi_0(z))}{\lambda} =: \varphi_1(z)$$

$F_2(\varphi_1)(z) := \varphi_2(z)$, where φ_2 is the unique solution of $\varphi_2(\lambda z) - \varphi_2(z) = \varphi_1(z)$.

$$F_3(\varphi_2)(z) := \int_0^z e^{\varphi_2(u)}\, du =: \varphi_3,$$

We look for a $\varphi = \varphi_0 = \varphi_3$, i.e., for a fixed point of $F_3 \circ F_2 \circ F_1$.

We must find appropriate function spaces where this transformation are well defined. For $g(z) = \sum_{n=1}^{\infty} a_n z^n \in z\,\mathbb{C}[[z]]$, define

$\|g\|_0 = \sum_{n=1}^{\infty} |a_n|$, $\|g\|_1 := \sum_{n=1}^{\infty} n|a_n|$

We set $H_\alpha = \{f : \mathbb{D} \to \mathbb{C},\ f(0) = 0,\ \|f\|_\alpha < +\infty\}$ $\alpha = 0, 1$.

Notice that $H_1 \subset H_0$.

H_α are banach spaces, and the convergence in H_α ~~corresponds to~~ implies uniform convergence on $\mathbb{D}$.

$\forall z, |z| \leq R < 1$

May consider $g \to 0$: then $|g(z)| \leq \sum |a_n||z|^n \leq R \cdot \sum |a_n| = R \cdot \|g\|_0$

Prop H_0 is a Banach algebra: $\|fg\|_0 \leq \|f\|_0 \|g\|_0$

H_1 is almost a Banach algebra: $\|fg\|_1 \leq 2\|f\|_1 \|g\|_1$ $\quad \|fg\|_\alpha \leq 2^\alpha \|f\|_\alpha \|g\|_\alpha$

Proof: $f = \sum a_n z^n$, $g = \sum b_n z^n \Rightarrow fg = \sum \left(\sum_{j=1}^{n-1} a_j b_{n-j}\right) z^n$

$\Rightarrow \|fg\|_0 = \sum \left| \sum_{j=1}^{n-1} a_j b_{n-j} \right| \leq \sum_n \sum_{j=1}^{n} |a_j||b_{n-j}| = \left(\sum_j |a_j|\right)\left(\sum_k |b_k|\right) = \|f\|_0 \cdot \|g\|_0$

$\|fg\|_1 = \sum n \left| \sum_{j=1}^{n-1} a_j b_{n-j} \right| \leq \sum_n n \sum |a_j||b_{n-j}|$

$\|f\|_1 \|g\|_1 = \sum_n \sum_{j=1}^{n-1} j(n-j)|a_j||b_{n-j}|$ notice that $\forall j = 1 \ldots n-1$, : $n \geq 2$
$n \leq 2j(n-j)$, $(n \leq 2n-2 \leq 2j(n-j))$

$\square$

Prop
It follows that: if $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a convergent power series with radius of convergence R, then $g \mapsto h \circ g$ is well defined as a map on $B_{H_0}(0, R)$ and on $B_{H_1}(0, \frac{R}{2})$ on $B_{H_\alpha}(0, \frac{R}{2^\alpha})$
$\to H_0$ $\qquad \to H_1$.

Proof: $h \circ g$ is well defined on $\mathbb{D}$ as far as $g(\mathbb{D}) \subseteq B(0, R)$

$|g(z)| \leq \sum |a_n z^n| < \sum |a_n| = \|g\|_0 < \|g\|_1$ Hence $h \circ g$ is well defined as far as $\|g\|_0 \leq R$.

If $h = \sum h_n z^n$, then:

$\| h \circ g \|_\alpha \leq \sum |h_n| \, \|g^n\|_\alpha \leq \sum |h_n| \cdot 2^{n\alpha} \|g\|_\alpha^n$, which converges

as far as $2^\alpha \|g\|_\alpha \lneq R$ □

Let now $U = \{ \Psi \in H_1 \, . \, \Psi'(0) = 1 \, , \, \| \Psi - \mathrm{Id} \|_0 \leq 1 \}$.

We want to prove that $F_3 \circ F_2 \circ F_1 : U \to U$. To do so, we may need to change coordinates linearly so that f has a big radius of convergence. i.e.:

$$f_m(z) = \frac{f(mz)}{m} = \lambda z + m a_2 z^2 + m^2 a_3 z^3 + \ldots$$

$$F_{1,m}(\psi) = \log\left(\frac{f_m'(\psi)}{\lambda} \right)$$

Prop: for $m \ll 1$, $F_{1,m}$ is defined on U, and sends U to an arbitrarily small neighborhood of 0 in H_1.

Proof: $F_{1,m}$ is defined on U as far as the radius of convergence of f_m is > 2. (By the previous prop.).

Moreover for $m \to 0$, $\|f_m\|_0 \to 0$, and $\|F_{1,m}(\psi)\|_1 \to 0$.

(From the fact that $\|h \circ g\|_1 \leq \|h\|_0$ as far as $\|g\|_1 < \frac{1}{2}$) □

We now deal with F_2

Prop: Let $g \in H_1$, and $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathcal{D}(2)$. There exist a constant C' depending only on α s.t. the equation:

$h(\lambda z) - h(z) = g(z)$ has a unique solution $h \in H_0$, and $\|h\|_0 \leq C' \|g\|_1$,

[where if C is s.t. $|\alpha - \frac{p}{q}| > \frac{C}{q^2}$, then $C' = \frac{\pi}{2C}$]

if $\alpha \in \mathcal{D}(d)$, $g \in H_{d-1} \rightsquigarrow \exists ! \, h \in H_0$, $\|h\|_0 \leq \frac{C_\pi}{2C} \|g\|_{d-1}$

Proof:

By developing everything in formal power series, we get:

$$\sum g_n z^n = \sum h_n \lambda^n z^n - \sum h_n z^n = \sum z^n (h_n(\lambda^n - 1))$$

From which, $h_n = \dfrac{g_n}{\lambda^n - 1}$

6.49

We now that $|\lambda^n - 1| = |e^{2\pi i \alpha n} - 1| \geq \frac{2}{\pi} \| \alpha n \| \geq \frac{2}{\pi} \frac{c}{n}$ $\quad (|\alpha - \frac{p}{q}| > \frac{c}{q^2})$

$\rightsquigarrow \|h\|_0 = \sum_{n \geq 1}^{\infty} |h_n| \leq \frac{\pi}{2C} \sum n \cdot |g_n| = \frac{\pi}{2C} \|g\|_1$ $\square$

Finally, we deal with F_3.

Prop: F_3 is a continuous map $H_0 \to H_1$ sending 0 to id

Proof: $F_3(\psi)(z) = \int_0^z e^{\psi(u)} du$.

$g \mapsto e^g - 1$ defines a continuous map from H_0 to H_0, $(z \mapsto e^z - 1 \in H_0$, with ∞ radius of convergence$)$. consider $g \mapsto e^g - 1 + 1 = e^g$.

The map $g \mapsto \underbrace{\int_0^z g(u) du}_{= h}$ is a continuous linear map $H_0 \to H_1$.

which is in fact an isometry; since $h(z) = \sum_{n \geq 1} \frac{g_{n-1}}{n} z^n$ $\quad (g(z) = \sum_{n \geq 0} g_n z^n)$.

Being F_3 the composition of this lin. maps, we get our statement $\square$.

Proof of the theorem:

For $m << 1$, the map $F_m = F_3 \circ F_2 \circ F_{1,m}$ maps U into itself, and in fact.

we may pick $m << 1$ so that $F_m(U) \subset U_r \subset U$, $U_r = \{\psi \in U \mid \|\psi - \mathrm{id}\|_0 \leq r < \}$.

Notice also that the map F_m is analytic (since for every step, the image depends analytically on the coefficients)

By (Earle)-Hamilton fixed point theorem, there exists a unique fixed point ψ for F, which is the solution of the linearization equation $\square$

Hamilton fixed point theorem: V Banach space, D open connected, $f: D \to D$ analytic, $f(D)$ bounded in norm, and $d(f(D), V \setminus D) > 0$, then $\exists!$ fixed point.

Idea of the proof: show that f is a contraction with respect to a distance opportunely defined (Carathéodory)

Bryuno's proof consists in studying very carefully the growth of the coefficients of the linearization Ψ, using "majorant series".

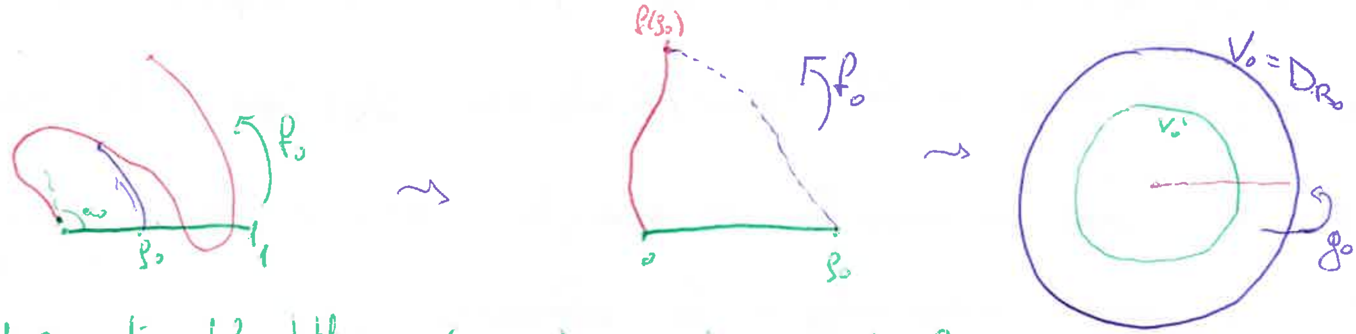
For a proof, see [Abate, local holomorphic dynamical systems]
which allows to prove the converse ($\alpha \notin B \Rightarrow \exists f, f'(0) = e^{2\pi i \alpha}, f \not\sim z \mapsto \lambda z$), and
Yoccoz gives a more geometric proof which also gives a more precise statement on the radius of convergence in terms of some quantities associated to α ($|\log \rho(\alpha) + B(\alpha)| \leq \underset{\text{universal constant}}{C}$ $\quad B(\alpha) = \sum_{n=1}^{\infty} \beta_{n-1} \log \frac{1}{\alpha_n}$)

One of the techniques introduced is called "Renormalization".

The idea is the following.

Let $\alpha_0 \in (0,1)$ ($\alpha_0 = \{\alpha\}$), and $f_0 : \mathbb{D} \to \mathbb{C}$ a univalent (injective) map with $f_0'(0) = e^{2\pi i \alpha_0}$.

The idea is to take $\rho_0 > 0$ small enough so that $f_0((0,\rho_0]) \cap (0,\rho_0] = \emptyset$.



Then take a sector U_0 between $(0,\rho_0)$ and its image by f_0. (= fundamental domain for f_0)

By quotienting U_0 by the action of f_0, we get $V_0 = U_0 / \langle f_0 \rangle \sim \mathbb{D}^*$.

The first return map to U_0 induces a holomorphic map $g_0 : V_0' \subset V_0 \to V_0$

We may renormalize V_0 so that $V_0' \supset \mathbb{D}$. g_0 extends to a map $\underset{\text{univalent}}{g_0} : \mathbb{D} \to \mathbb{C}$

Notice that a rotation of angle α_0 on U_0 correspond to a rotation of angle 1 on V_0.

It follows that $g_0'(0) = e^{-2\pi i \alpha_1}$, where $\alpha_1 = \{\frac{1}{\alpha_0}\}$ $(-\alpha_1 = 1 - \{\frac{1}{\alpha_0}\})$

Set $f_1(z) = \overline{g_0(\bar{z})}$, which has multiplier α_1.

~~The~~ We can now iterate the procedure, and construct a sequence of renormalized maps f_n of multiplier α_n.

The estimate of the radius of the Siegel disc depends on very precise estimates of the numbers β_n. [Buff-Hubbard for further details].

Once achieved the Brjuno characterisation of linearizability, a natural question is to understand what happens when $\alpha \notin \mathcal{B}$ and f is not linearizable.

Using techniques similar to Yoccoz's renormalization, Perez-Marco obtained the following result:

Def: $\alpha \in [0,1)$ satisfies the Perez-Marco condition if $\sum_{n\geq 1} \frac{\log \log q_{n+1}}{q_n} < +\infty$

$\left(\Leftrightarrow \sum_{n\geq 0} \beta_{n-1} \cdot \log \log \frac{e}{\alpha_n} < +\infty\right)$ Notice: $\mathcal{B} \subset \mathcal{PM} = \{\text{Perez-Marco numbers}\}$.

~~Theorem (Perez-Marco)~~: Perez-Marco condition $\leftrightarrow$ rationals approximate α not too fast (but maybe faster than $\mathcal{B}$).

Theorem (Perez-Marco): Let $\alpha \in \mathcal{PM}$. $f:(\mathbb{C},0)\circlearrowleft$ so that $f'(0)=e^{2\pi i\alpha}$. Then

Either • f is linearizable, or

• f has small cycles.

Def: $f:(\mathbb{C},0)\circlearrowleft$ has small cycles if any neighborhood of $\{0\}\subset\mathbb{C}$ contains a periodic orbit for f.

Rem: By modifying Pfeiffer's argument one can construct examples of irrational germs with small cycles, see [Buff-Hubbard, Theorem (5.)6.1]

If $\alpha \notin \mathcal{PM}$ (α is very well approximated by rationals), then ~~we may~~ $\exists f$ that is not linearizable, and does not admit small cycles.

Rem: it is not known if this can happen for the germ of a rational map in $\mathbb{P}^1$.

In his work, Perez-Marco also describes the structure of the stable set.

Recall that $K_f(U)=\bigcap_{n\in\mathbb{N}} f^{-n}(U)$. If f is linearizable, then $K_f(U)$ is a neighborhood of the origin, and there exist a continuous family of totally invariant compacts (corresponding to $\overline{D_r}$ in the linear model).

Moreover, any point in K_f is recurrent (i.e., any point belongs to the adherence of its orbit: $\forall p, \forall \varepsilon>0\ \exists n\in\mathbb{N}$ s.t. $f^n(p)\in D(p,\varepsilon)$).

A similar situation occurs in general, but with a set K_f much more complicated.

Thm (Perez-Marco) Let $f:(\mathbb{C},0)\circlearrowleft$ be an irrational Cremer germ (i.e., non linearizable).

Then K_f is ~~totally~~ compact, connected, full ($\mathbb{C}\setminus K_f$ is connected), not reduced to $\{0\}$, with empty interior, not locally connected at any point in $K_f\setminus\{0\}$.

- ~~any point~~ in K_f is encircled by totally invariant compacts with the same properties K_r.
- Any point of $K_f\setminus\{0\}$ is recurrent.
- There is an orbit that accumulates at the origin, but no non-trivial orbit converges to 0.

Thm (Biswas) The rotation number α and the conformal class of K_f are a complete invariant of holomorphic conjugacy.

The K_r as above are called hedgehog associated to f.

Idea: $\alpha \sim \frac{p_n}{q_n}$: parabolic germ $f_n = e^{2\pi i \frac{p_n}{q_n}} z + g(z)$, $f=\lambda z+g(z)$, $g(z)=o(z)$